Return current in encephalography

Variational principles

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ABSTRACT The encephalographic problem of finding the electric potential V and the return current associated with any assumed primary current, $\mathbf{J}^p$, is put in the form of a variational principle. With $\mathbf{J}^p$ and the conductivity specified, the correct V is one which makes an integral quantity $P[V]$ a maximum. The terms in $P[V]$ are related to the rates at which work is done by the electric field on the primary and return currents. It is shown that there is a unique solution for the electric field, and it satisfies the conservation of energy; this condition can serve as a check on any numerical solution.

With the conductivity a different constant in different regions, the variational principle is recast in terms of the charge density on the surfaces of discontinuity. An iteration-variation method for finding the solution is outlined, and possible computational advantages over other approaches are discussed.

INTRODUCTION

The total electric current $\mathbf{J}$ within the brain has been written as a sum of two terms of distinctly different nature (Geselowitz, 1967; Barnard et al., 1967). The first, $\mathbf{J}^p$, is the "primary" current that flows within neurons, and is the quantity of interest in neuroscience. This current leaks across the cell membrane, however, and because the cells are embedded in an electrically conducting medium, the extracellular ("return") current flows through a large volume of this medium before returning to the cell. With the return current taken to be the product of the local conductivity σ and the electric field intensity $\mathbf{E}$, the complete current becomes

$$\mathbf{J} = \mathbf{J}^p + \sigma\mathbf{E}. \tag{1.1}$$

Examination of the continuity equation,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \tag{1.2}$$

together with Gauss's law,

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \tag{1.3}$$

shows that $\partial\rho/\partial t$ can be neglected to a good approximation provided the time interval during which $\mathbf{J}^p$ undergoes appreciable change is large compared with ϵ_0/σ. Because even the skull has the rather small value $\epsilon_0/\sigma \simeq (200$ ohm-m) $\epsilon_0 = 1.8 \times 10^{-9}$ s (Nunez, 1981), this approximation leads to the equation (Geselowitz, 1967),

$$\nabla \cdot (\mathbf{J}^p + \sigma\mathbf{E}) = 0. \tag{1.4}$$

If the medium carrying the return current has a dielectric constant κ different from unity, then inclusion of the polarization current in Eq. 1.1 changes the time constant to $\kappa\epsilon_0/\sigma$. Although we are not aware of an experimental value of κ for the skull, it is not likely to be so large as to change the conclusion.

In both electro- and magnetoencephalography the "forward" problem (which is the only one discussed in this paper) consists of making a simple assumption about $\mathbf{J}^p$, e.g., a point current dipole at some position, and comparing the computed electric or magnetic field with that measured in an experiment. Some of these analyses completely neglect the contribution of the return current; while this is satisfactory for the normal component of the magnetic field if the geometry is nearly spherically symmetric (Sarvas, 1987), there are conditions in which this approximation is not accurate (Meijs et al., 1987). The attempts that have been made to include the return current have taken the conductivity to be a different constant in each of a number of regions, e.g., brain, skull, and scalp. Analytic solutions of Eq. 1.4 have been obtained for a single conducting spherical or spheroidal region (Cuffin and Cohen, 1977; Sarvas, 1987). For more general geometries Eq. 1.4 has been solved by direct numerical methods (Witwer et al., 1972) and also by converting it to a surface integral equation for the electric potential V (Geselowitz, 1967), and then solving that equation numerically (Meijs et al., 1987; Hämäläinen and Sarvas, 1989). For the sake of completeness we record this integral equation,

$$\frac{\sigma' + \sigma''}{2} V(\mathbf{r}) = \frac{1}{4\pi} \int d^3r' \mathbf{J}^p(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi} \sum_j (\sigma_j' - \sigma_j'') \int dS\mathbf{n}(\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} V(\mathbf{r}'), \quad (1.5)$$

where $\mathbf{r}$ is a point on one of the surfaces where the conductivity jumps in value from σ' to σ'', and the sum is over all these surfaces.

In this paper we present a different method for solving Eq. 1.4 for the return current, based on a variational principle that is described in Section II. It is shown in Section III that this principle involves the rates at which the electric field does work on the primary and return currents, and furthermore, that the correct solution satisfies conservation of energy. In Section IV the variational principle is rewritten to allow for point sources, and in Section V the special case is considered in which the conductivity is a different constant in different regions. An iteration-variation procedure for finding the solution is outlined in Section VI, and numerical methods are discussed in Section VII.

II. THE VARIATIONAL PRINCIPLE

Having made a model of the conductivity σ throughout the head, and an assumption about the primary current $\mathbf{J}^p$, these quantities are now taken as known. The variational principle states that the correct electric potential V *minimizes* the rate of dissipation of energy in the extracellular medium, but subject to the continuity condition, Eq. 1.4. With this *constraint* included via a Lagrange multiplier, the variational principle can be manipulated into the following statement. The solution of Eq. 1.4 for the electric potential is that function $V(\mathbf{r})$, which makes the quantity $P[V]$ an extremum, where

$$P[V] = \int d^3r[2D^pV - \sigma(\nabla V)^2], \quad (2.1)$$

and for ease of writing we have defined

$$D^p = -\nabla \cdot \mathbf{J}^p. \quad (2.2)$$

(It is clear from Eq. 1.4 that it is only the divergence of $\mathbf{J}^p$ that is relevant to the solution for $\mathbf{E}$.) The desired electric field is then obtained from

$$\mathbf{E} = -\nabla V. \quad (2.3)$$

After proving the assertion made above, we will show that P is in fact a *maximum*, rather than a saddle point or minimum, at the correct V. (Because of the sign we have chosen in Eq. 2.1, this represents a *minimum* rate of dissipation of energy. See Section III.)

Consider the first order change δP that is produced by a variation in the potential function δV,

$$\delta P = 2 \int d^3r[D^p\delta V - \sigma\nabla V \cdot \nabla\delta V], \quad (2.4)$$

and use the identity

$$\nabla \cdot (X\mathbf{A}) = X\nabla \cdot \mathbf{A} + \nabla X \cdot \mathbf{A} \quad (2.5)$$

to obtain

$$\delta P = 2 \int d^3r[D^p + \nabla \cdot (\sigma\nabla V)]\delta V - 2 \int dS[\mathbf{n} \cdot \sigma\nabla V]\delta V. \quad (2.6)$$

The second integral in Eq. 2.6 is over the area of the surface that bounds the volume, and $\mathbf{n}$ is a unit vector normal (outward) to that surface.

In order that δP vanish for an *arbitrary* change δV, it is necessary that *both* bracketed factors in the integrands of Eq. 2.6 be zero. The first one leads to

$$\nabla \cdot (\sigma\nabla V) = -D^p \quad (2.7)$$

at every interior point, and the second leads to

$$\mathbf{n} \cdot \sigma\nabla V = 0 \quad (2.8)$$

at every point on the surface. Eq. 2.7 is precisely the desired equation (1.4), once Eqs. 2.2 and 2.3 are recalled, thereby proving the claim that the correct electric potential makes P (Eq. 2.1) an extremum. Eq. 2.8 is equally valid for $\sigma\nabla V$ just inside or just outside the surface because it states that no current crosses the surface; it is the appropriate boundary condition because σ vanishes outside the head. This completes the proof that an extremum of $P[V]$ provides a solution to the problem.

It is straightforward to show that an extremum of P is, in fact, a maximum and, furthermore, that the solution for ∇V is unique. Suppose that some function V makes P an extremum, and again consider the effect of a variation δV upon P. This time, however, we shall write the exact result, not just the first order change,

$$P[V + \delta V] = \int d^3r\{2D^p(V + \delta V) - \sigma[\nabla(V + \delta V)]^2\} = P[V] - \int d^3r\sigma(\nabla\delta V)^2. \quad (2.11)$$

The final line of Eq. 2.11 follows from the fact that the first order change vanishes at an extremum, and because P is only of second order in V, there are no higher order terms in δV.

Equation 2.11 shows that for arbitrary δV,

$$P[V + \delta V] \leq P[V] \quad (2.12)$$

if V is a solution. Furthermore, only δV = constant leads to the equality sign in Eq. 2.12; changing V by a constant, however, has no physical significance. This shows that the solution is a *maximum* of P. It also shows that there can only be one solution (apart from an additive constant) because the two values of P at two supposed solutions would each have to be larger than the other.

Before extracting additional results from Eq. 2.1 we note that a variational principle has also been used in electrocardiography (Yamashita and Takahashi, 1984; Pilkington et al., 1985), but which differs from Eq. 2.1 in two important respects. In the ECG problem one tries to relate the potential on the surface of the heart to that measured on the body surface, and it is assumed that there are no current sources in the volume between these surfaces. The term linear in V in Eq. 2.1 is not present, therefore.

The other difference between ECG and encephalography is that the boundary condition on the heart surface is not given by Eq. 2.8 because current can flow across that surface. Instead, the values of the potential there are taken as a constraint upon the allowed functions V. The trial functions V in Eq. 2.1, on the other hand, are not constrained.

III. CONSERVATION OF ENERGY

Suppose that the electric potential V that makes P (Eq. 2.1) a maximum, and which therefore represents the solution of Eq. 1.4 via Eq. 2.3, has already been found. One special variation of P consists of multiplying this V by an overall constant N, giving

$$P[NV] = \int d^3r[2D^pNV - \sigma N^2(\nabla V)^2]. \tag{3.1}$$

Now vary N to obtain

$$\delta P = 2\int d^3r\,[D^pV - \sigma N(\nabla V)^2]\delta N. \tag{3.2}$$

But δP must vanish for $N = 1$ because that is the correct solution we started with. Because the two terms in Eq. 3.2 are equal to each other with $N = 1$, at the correct solution $P[V]$ has the value

$$P_0[V] = \int d^3rD^pV - \int d^3r\sigma(\nabla V)^2. \tag{3.3}$$

We shall now show that the first integral in Eq. 3.3 represents the rate at which the neurons do work in creating the current $\mathbf{J}^p$ (in the presence of the electric field $\mathbf{E} = -\nabla V$); and the second integral in Eq. 3.3 is the rate at which electrical energy is dissipated (ohmic heating). Therefore, this equation represents the *conservation of energy*. To see this, recall that for any current $\mathbf{J}$, $\int d^3r\mathbf{J}\cdot\mathbf{E}$ represents the rate at which the electric field does work *on* the current. It follows from the definition of D^p (Eq. 2.2) that

$$\int d^3rD^pV = -\int d^3rV\nabla\cdot\mathbf{J}^p = -\int d^3r\mathbf{J}^p\cdot\mathbf{E} \tag{3.4}$$

with the last step following from the identity (Eq. 2.5). The minus sign in the final version of Eq. (3.4) shows that $\int d^3rD^pV$ does indeed represent the rate at which the neuronal current does work against the electric field. (The middle version of that equation has an interpretation completely consistent with this. If the neuronal current were the only one present, the continuity Eq. 1.2 would make that quantity equal to $\int d^3rV\partial\rho^p/\partial t$. But this is precisely the rate at which the neurons would do work to create a changing charge density $\partial\rho^p/\partial t$ at a point where the electric potential has the value V.)

The final form of Eq. 3.3, $\int d^3r(\sigma\mathbf{E})\cdot\mathbf{E}$, represents the rate at which the electric field does work on the return current, this energy appearing as heat. Eqs. 3.3 and 3.4 demonstrate that energy is conserved overall. They can be combined to read

$$\int d^3r\mathbf{J}\cdot\mathbf{E} = \int d^3r(\mathbf{J}^p + \sigma\mathbf{E})\cdot\mathbf{E} = 0. \tag{3.5}$$

Because the correct solution to Eq. 1.4 must satisfy Eq. 3.3 (or Eq. 3.5, which is the same thing), it can serve as a valuable check on any approximate numerical solution, no matter what technique is used to obtain it.

IV. POINT SOURCES

Eq. 2.1 is not suitable for numerical work if the divergence of the primary current is concentrated at individual points, which is the case with a current dipole, for example, because near such a point $\int d^3r(\nabla V)^2$ diverges. To overcome this problem write V as a sum of a known term V^p and an (as yet) unknown term ϕ,

$$V = V^p + \phi, \tag{4.1}$$

where V^p is given by

$$V^p(\mathbf{r}) = \frac{1}{4\pi}\int d^3\mathbf{r}'\frac{D^p(\mathbf{r}')}{\sigma(\mathbf{r}')|\mathbf{r} - \mathbf{r}'|}, \tag{4.2}$$

and therefore satisfies the equation

$$\nabla^2V^p = -\frac{D^p}{\sigma}. \tag{4.3}$$

Inserting Eq. 4.1 into Eq. 2.1 leads to

$$P[V] = P^p + p[\phi], \tag{4.4}$$

where

$$P^p = \int d^3r[2D^pV^p - \sigma(\nabla V^p)^2], \tag{4.5}$$

and $p[\phi]$ contains all the dependence on the unknown function ϕ,

$$p[\phi] = \int d^3r[2D^p\phi + 2\sigma\mathbf{E}^p\cdot\nabla\phi - \sigma(\nabla\phi)^2]. \tag{4.6}$$

$\mathbf{E}^p$ has been defined to be

$$\mathbf{E}^p = -\nabla V^p \tag{4.7}$$

Because V^p is completely specified by the conductivity function and the assumed primary current (Eq. 4.2), p^p plays no role in the variational principle, which reduces to finding the function ϕ that makes $p[\phi]$ in Eq. 4.6 a maximum.

Two alternative ways of writing $p[\phi]$ are obtained from Eqs. 4.3 and 2.5, as

$$p[\phi] = \int d^3r\sigma[2\nabla \cdot (\phi\mathbf{E}^p) - (\nabla\phi)^2], \tag{4.8}$$

$$p[\phi] = -\int d^3r[2\phi\nabla\sigma \cdot \mathbf{E}^p + \sigma(\nabla\phi)^2]. \tag{4.9}$$

Note that the primary current enters Eq. 4.9 (via $\mathbf{E}^p$) only at points where the conductivity is changing.

V. REGIONS OF CONSTANT CONDUCTIVITY

In the previous section the decomposition of V into $V^p + \phi$ was made because the original form of the variational principle, Eq. 2.1, does not converge if there are points where the primary current begins or ends. Whether or not there are point sources, if it is a good assumption to say that the conductivity is a different constant in different regions, e.g., brain, skull, and scalp, then this decomposition has a direct physical interpretation. This can be seen from Eq. 2.7. Inside any region of constant σ, that equation becomes

$$\nabla^2 V = -\frac{D^p}{\sigma}, \tag{5.1}$$

and comparison with Eq. 4.3 shows that

$$\nabla^2\phi = 0 \text{ (inside each region).} \tag{5.2}$$

At the boundary between two regions having conductivities σ' and σ'', Eq. 2.7 becomes

$$\mathbf{n} \cdot (\sigma'\nabla'V - \sigma''\nabla''V) = 0, \tag{5.3}$$

assuming that no primary current begins or ends on the surface.

The significance of Eqs. 5.1–5.3 is as follows. Wherever the primary current starts or stops, and consequently $D^p = -\nabla \cdot \mathbf{J}^p$ is not zero, electric charge density equal to $\epsilon_0 D^p/\sigma$ appears. This charge creates an electric potential V^p and an electric field $\mathbf{E}^p = -\nabla V^p$, which in turn produces *return* current. If the conductivity were uniform throughout all space, the return current would be precisely equal to $\sigma\mathbf{E}^p$, and ϕ would be zero everywhere. (This has been called the "maximal" current [Heller, 1972].) It is straightforward to show that this particular return current does not produce any magnetic field whatsoever (Tripp, 1983), so the entire magnetic field would be due to the primary current $\mathbf{J}^p$.)

On a surface where the conductivity jumps in value, however, electric charge appears to insure that the current is continuous across the surface (Eq. 5.3). This surface charge density is the source of the potential ϕ, whereas the volume charge density $\epsilon_0 D^p/\sigma$ is the source of the potential V^p. Denoting the constant values of the conductivity by σ_j, it follows from Eq. 4.8 or 4.9 that

$$p[\phi] = -\sum_j \sigma_j \int d^3r(\nabla\phi)^2 + 2\sum_j (\sigma_j' - \sigma_j'') \int dS\mathbf{n} \cdot \mathbf{E}^p\phi. \tag{5.4}$$

We have followed the convention that the unit normal vector $\mathbf{n}$ points *outward* from the region with conductivity σ' and *into* the region with conductivity σ'', and the second term in Eq. 5.4 is summed over all the surfaces on which σ is discontinuous. Using Eq. 2.5 on the first term of Eq. 5.4 yields another variant,

$$p[\phi] = \sum_j \sigma_j \int d^3r\phi\nabla^2\phi + \sum_j \int dS\mathbf{n} \cdot [2(\sigma_j' - \sigma_j'')\mathbf{E}^p - (\sigma_j'\nabla'\phi - \sigma_j''\nabla''\phi)]\phi. \tag{5.5}$$

In trying to maximize $p[\phi]$ in any of the Eqs. 4.6, 4.8, 4.9, 5.4, or 5.5, there are no restrictions on the trial function ϕ. We know, however, from Eq. 5.2 that the sources of the *correct* function ϕ are all on the surfaces of discontinuity of σ, so it is sensible to parameterize the variational function ϕ in terms of a surface charge density τ,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_j \int dS' \frac{\tau(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{5.6a}$$

Because Eq. 5.6a satisfies Eq. 5.2 inside every region for *any* choice of the function τ, the first term in Eq. 5.5 can now be set equal to zero:

$$p[\phi] = \sum_j \int dS\mathbf{n} \cdot [2(\sigma_j' - \sigma_j'')\mathbf{E}^p - (\sigma_j'\nabla'\phi - \sigma_j''\nabla''\phi)]\phi. \tag{5.6b}$$

The problem has been reduced to finding the surface charge τ which, when used in Eq. 5.6a, makes Eq. 5.6b a maximum. Repeating the argument from Eqs. 3.1–3.3 shows that at the correct solution $p[\phi]$ has the value

$$p_0[\phi] = \sum_j (\sigma_j' - \sigma_j'') \int dS\mathbf{n} \cdot \mathbf{E}^p\phi = \sum_j \int dS\mathbf{n} \cdot (\sigma_j'\nabla'\phi - \sigma_j''\nabla''\phi)\phi. \tag{5.7}$$

VI. ITERATION-VARIATION PROCEDURE

We now consider the question of how to make a first guess for τ, and then how to improve that guess. For this purpose it is useful to decompose the electric field due to the surface charge into two terms. One of them, $\mathbf{E}^s(\mathbf{r})$, is the field at the surface point $\mathbf{r}$ arising from all the surface charge *except* the singular contribution due to the charge at $\mathbf{r}$ itself,

$$\mathbf{E}^s(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int dS'\tau(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \quad (\mathbf{r}\text{ on the surface}). \quad (6.1)$$

The surface charge right at $\mathbf{r}$ makes a contribution to the normal component of the electric field of magnitude $\tau(\mathbf{r})/2\epsilon_0$, which points *away* from the surface. Putting the two terms together gives, on the two sides of the surface,

$$-\mathbf{n}\cdot\nabla''\phi = \mathbf{n}\cdot\mathbf{E}^s + \frac{1}{2\epsilon_0}\tau$$

and

$$-\mathbf{n}\cdot\nabla'\phi = \mathbf{n}\cdot\mathbf{E}^s - \frac{1}{2\epsilon_0}\tau. \quad (6.2)$$

Making use of Eqs. 4.1, 4.7, and 6.2, continuity of the current, Eq. 5.3, then leads to

$$\sigma''[\mathbf{n}\cdot(\mathbf{E}^p+\mathbf{E}^s) + \frac{1}{2\epsilon_0}\tau] = \sigma'[\mathbf{n}\cdot(\mathbf{E}^p+\mathbf{E}^s) - \frac{1}{2\epsilon_0}\tau], \quad (6.3)$$

or

$$\tau(\mathbf{r}) = 2\epsilon_0\frac{\sigma'-\sigma''}{\sigma'+\sigma''}\mathbf{n}(\mathbf{r})\cdot[\mathbf{E}^p(\mathbf{r})+\mathbf{E}^s(\mathbf{r})]. \quad (6.4)$$

Eqs. 6.1 and 6.4 represent a two-dimensional integral equation for the surface charge density τ (Gelernter and Swihart, 1964; Rush et al., 1966; Barnard et al., 1967). One could undertake a direct numerical solution of this equation in a manner similar to that for the potential V on the surface, Eq. 1.5. It is also possible to attempt a solution by iteration (Gelernter and Swihart, 1964) but it is not clear that this procedure will always converge. Another procedure, which systematically approaches the correct solution, is to treat the integral equation for τ together with the variational principle via a *combined* iteration-variation technique, as follows.

Suppose at the mth stage one has an approximation τ_m to τ, and a corresponding value of $p[\phi_m]$ from Eqs. 5.6. Inserting τ_m into Eq. 6.1, and using that function $\mathbf{E}^s_m$ on the right side of Eq. 6.4 gives a quantity that we designate $\tau_{m+1/2}$. By forming the linear combination

$$\tau_{m+1} = \alpha\tau_m + \beta\tau_{m+1/2}, \quad (6.5)$$

and putting this into Eqs. 5.6, a value for $p[\phi_{m+1}]$ is obtained, which is then maximized with respect to α and β. It is clear that $p[\phi_{m+1}]$ will exceed $p[\phi_m]$ in value, and therefore τ_{m+1} is a better approximation than τ_m. This process is then repeated. (In the unlikely event that $\alpha = 1$ and $\beta = 0$, a new guess for $\tau_{m+1/2}$ must be made.) An obvious first guess for τ is obtained by putting $\mathbf{E}^s = 0$ on the right side of Eq. 6.4. It is worth noting that for the case of a single infinite plane surface, this choice of τ is exact.

Once the electric potential has been found, the magnetic field is obtained from (Geselowitz, 1970):

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\mathbf{J}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$
$$= \frac{\mu_0}{4\pi}\left(\int d^3r'\mathbf{J}^p(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \sum_j(\sigma_j'-\sigma_j'')\int dS'V(\mathbf{r}')\mathbf{n}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right), \quad (6.6)$$

where the first term in brackets is the field due to the primary current and the second term arises from the return current.

VII. NUMERICAL METHODS

A number of different mathematical procedures are available for solving the return current problem, including the three-dimensional partial differential Eq. 1.4, the two-dimensional integral Eqs. 1.5 and 6.4, and the variational principles presented in this paper. Using a finite difference scheme on the differential equation (Witwer et al., 1972) leads to a system of simultaneous linear equations. If it is desired to model the primary current $\mathbf{J}^p$ with fine spatial resolution then this would require a very large number of grid points, which would make the dimension of the matrix very large. Tessellation of the surfaces for the integral equation (Meijs et al., 1987; Hämäläinen and Sarvas, 1989) also leads to a system of linear equations, which would become large if more and more regions having different conductivities were considered.

Any version of the variational principle for $p[\phi]$ can also be converted to a set of linear equations, by the method of finite elements. This is true for the three-dimensional versions, Eqs. 4.6, 4.8, and 4.9; the mixed two- and three-dimensional versions, Eqs. 5.4 and 5.5; and the constrained two-dimensional version, Eq. 5.6.

The iteration-variation procedure described in Section VI has the advantage that one does not have to solve a system of equations; it is only necessary to do the indicated integrations. The key question is how many iterations will be required to get convergence. We suspect that this method may be significantly faster than the others, and intend to study this question for a variety of assumed primary currents $\mathbf{J}^p$ and conductivity functions σ.

SUMMARY

The problem of computing the return current $\sigma\mathbf{E}$ associated with an assumed primary current $\mathbf{J}^p$ has been treated previously by direct numerical solution of Eq. 1.4, and also by solving integral Eq. 1.5 for the electric potential V on the surfaces which separate regions of different conductivity. In this paper we have presented alternative methods for computing V via a variational principle, which are suitable for an arbitrary conductivity function; we have also specialized to the case in which σ is a different constant in different regions. For the latter case the problem is to find the distribution of electric charge τ on the surfaces of discontinuity of σ such that the electric potential ϕ which it produces makes the variational quantity $p[\phi]$ a maximum. This result is contained in Eq. 5.6. The complete electric potential is given by $V = V^p + \phi$, where V^p is determined by the volume charges, Eqs. 4.2 and 2.2, and ϕ is determined by the surface charges.

We have shown that once the correct ϕ, and therefore V, has been found, the solution satisfies conservation of energy. This result is embodied in Eq. 3.3 (or 3.5, which is equivalent). Any approximate numerical solution of the return current problem, no matter how it is obtained, ought to be tested to see how well it satisfies this condition.

Because $p[\phi]$ has only one maximum, any change in ϕ that increases p is a step in the right direction. In section VI we have described an iteration-variation procedure for making a first guess for τ and then systematically improving upon it. We expect to develop this into an actual numerical procedure and apply it to realistic models of the head.

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